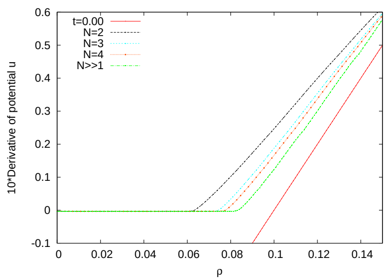
Results

The first result we achived is the combination of FRG flow equation and KT scheme. This has been done for the $O(N)$ model, using the Local potential approximation as an ansatz for the bare action: this means that we stopped the operator expansion for the effective action at the 0^{th} order, including only the effective potential. This will transform the RG flow equation into a partial differential equation for the effective potential, which can be cast in the hydrodynamic form of a convection diffusion equation.

We then investigated the $O(N)$ model both for $N \gg 1$ and for finite N. In the $N \gg 1$ case we studied, varying the initial conditions for the bare action, how the phase transitions develop as the RG time flows increases (which means as the quantum fluctuations are integrated). In particular we studied three cases, corresponding to three different initial conditions, to test the accuracy of our method: the Riemann problem, which corresponds to a discontinuous initial data, a quadratic potential in the fields expectation value, which leads to a second order phase transition, and a cubic potential, which originates a first order phase transition. We then examined the different kind of phase transition extrapolating the asymptotic behavior (RG time $t \to \infty$, which corresponds to have integrated all the fluctuations) of the effective potential and its derivative, and obtained the critical exponents.

We than analyzed the finite N case and the impact of the related diffusion flux on the typology of phase transition. We tested it both in the case of the Riemann problem and in the second order phase transition.



In particular we found that the diffusion term helps the stability of the numerical scheme and does not modify the quality of the phase transition. As an example we report in figure the result we obtained, varying the number of field N, in the case of the second order phase transition, for the derivative of the effective potential.

Conclusions

In this work we have seen that the Functional Renormalization Group is one of the most powerful tools that have been developed in order to study Quantum Field Theories. It allows, in theory, to compute the full quantum effective action in a non-perturbative way, even if exact solutions are rare and numerical methods are needed. The highly innovating charge of this work consists in the application, for the first time, of an Hydrodynamic algorithm, like KT scheme, to the FRG flow equation, which allows to treat FRG flow equation as a standard convection diffusion equation. We have applied this method to the $O(N)$ model, studying phase transitions both in the finite N and $N \gg 1$ case. However a great variety of extension, such us adding the quark degrees of freedom or generalizing the theory to finite temperature, can be taken into account and solved, as far as they concern the addiction of a term which has the structure, even if not trivially, of a convection flux or a source term. Thanks to KT scheme high accuracy, non-oscillatory properties and versatility it will be possible to improve the current results and to obtain a wide range of new results, applying the scheme in a straightforward way. Furthermore this approach can be applied not only, as previously described, to the effective field theories and models like $O(N)$, but also, in principle, to any problem whose description can be modeled by an equation of the convection-diffusion type, in the field of physics and beyond, showing how powerful and versatile this tool is.

Acknowledgements

I thank both prof. Greco and prof. Rischke for their help and for their time.

Contact details

Fabrizio Murgana `fabrizio.murgana@studium.unict.it`

UNIVERSITÀ degli STUDI di CATANIA

Dipartimento di Fisica e Astronomia *"Ettore Majorana"*

MSc PROGRAMME IN PHYSICS

FABRIZIO MURGANA

HYDRODYNAMIC APPROACH TO FUNCTIONAL RENORMALIZATION GROUP

FINAL PROJECT

SUPERVISORS:
CHIAR.MO PROF. V. GRECO
CHIAR.MO PROF. D. RISCHKE

ANNO ACCADEMICO 2019/2020

Abstract

The Functional Renormalization Group (FRG) approach to Effective field theory is a more recent and very powerful tool. In the context of the physics of strong interactions, however, it is very hard to find exact analytical solutions and thus the development of sophisticated numerical methods are needed. Until now only Finite Difference schemes have been used, even though they don't provide a sharp resolution of the shock waves and discontinuities that develop in the solutions of the FRG flow equation. On the other hand Kurganov and Tadmor (KT) scheme, a new class of central schemes, was developed as a tool for hydrodynamic equations, and thus combines shock resolution and accuracy. In this work we applied this scheme to solve numerically the flow equations that derive from the $O(N)$ model, both in the finite N and $N \gg 1$ case, going beyond mean field approximations, and studying the associated phase transitions.

Background

The problem of solving FRG flow equation has been treated both analytically, with approximations, and numerically. From the numerical point of view, studies of the flow equations of effective field theories and their generalizations have been done, but mostly in the framework of finite-difference schemes, with the severe problem of lacking shock resolution. A first attempt of using a hydrodynamic finite-volume scheme has been made by E. Grossi and N. Wink using the discontinuous Galerkin method. However, the results are not of immediate use, since they need a smearing through a limiter to eliminate Gibbs oscillations. Furthermore, these studies are limited to the $O(N)$ scalar model, in the local potential approximation, and to the case of large N. Another aspect to underline is the power of the FRG approach in the study of phase transitions. In fact, until now many models and effective field theories, including the $O(N)$ scalar model, which can be considered as a prototype model for phase transitions, have been solved analytically, but in the framework of the mean-field approximation. However, this approximation is not sufficient in the description of a system close to criticality, since it neglects the thermal and/or quantum fluctuations that become relevant and dominant dealing with a phase transition. This means that it is mandatory to go beyond the mean-field approximation. An optimal choice is thus to use the FRG approach, since once the flow equation has been solved and the full quantum action has been computed, it is able to take into account all the fluctuations, leading to a correct detection and analysis of a phase transition.

Objectives

The main goal of this work is the development of a new kind of approach to the problem of the solution of FRG flow equation. In particular it is hardly possible to solve this equation analytically, and thus a sophisticated and efficient numerical tool needs to be developed. In order to do this the first thing to do is to derive the flow equation for the studied quantum field theory. Once this has been done, the key target is to put the flow equation into a conservative form, i.e. in the form of a conservation-diffusion equation, which is typical of hydrodynamic problems. This will allow us to use hydrodynamic kind of algorithms, as KT scheme, which is able to give a sharp and precise resolution of the shocks and rarefactions that often appear in the solutions of FRG flow equation.

In particular we want to study of the $O(N)$ model through the functional renormalization group technique. This means that we are interested in the investigation of the phase structure and the related phase transitions for the $O(N)$ model, both in the $N \gg 1$ case, which is the most studied case, and in the finite N case. This last case is of relevant interest since it introduces, with respect to the large N case, the presence of a diffusion flux. Thus one of the points is to study if and how the presence of such a diffusion term modifies the phase structure of the problem.

Techniques

The first technique we used is the functional renormalization group approach to a quantum field theory. In particular functional methods aim at the computation of generating functionals of correlation functions, such as the effective action that governs the dynamics of the macroscopic expectation values of the fields. These generating functionals contain all relevant physical information about a theory, once the microscopic fluctuations have been integrated out. The functional RG combines this functional approach with the RG idea of treating the fluctuations not all at once but successively from scale to scale. Instead of studying correlation functions after having averaged over all fluctuations, only the change of the correlation functions as induced by an infinitesimal momentum shell of fluctuations is considered. From a structural viewpoint, this allows to transform the functional-integral structure of standard field theory formulations into a functional differential structure.

The central tool of the functional RG is given by a flow equation which describes the evolution of correlation functions or their generating functional under the influence of fluctuations. It connects a well-defined initial quantity, e.g., the microscopic correlation functions in a perturbative domain, in an exact manner with the desired full correlation functions after having integrated out the fluctuations. Hence, solving the flow equation corresponds to solve the full theory. Since it allows to take into account all the fluctuations, the FRG approach seems suitable to deal with the physics of phase transitions, when the fluctuations become dominant.

The second main tool used is the Kurganov and Tadmor scheme, which belongs to the class of central schemes. In general central schemes are finite-volume methods for solving non-linear convection–diffusion equations. Their main advantage is their simplicity since they are not tied to the specific eigenstructure of the problem and thus can be implemented as a solvers for general conservation laws and related equations.

In particular the Kurganov-Tadmor central scheme retains the simplicity of being independent of the eigenstructure of the problem, yet gains a much smaller numerical viscosity with respect to the forerunners of central schemes: LxF scheme and NT scheme. In particular, Kurganov-Tadmor scheme maintains a high-resolution of shocks and rarefactions independently of the time spacing Δt, and letting $\Delta t \to 0$ it admits a particularly simple semi-discrete formulation. The main advantage of this formulation is the possibility to modify Δt and make it as small as it is needed in order to prevent the formation of spurious oscillations in the solution of the problem.